

REAL ANALYSIS 1

Sets and Functions. First we'll define the notation of a set.

Definition 1 (Set). A *set* is a collection of objects, called *elements* (or *members*). We denote that an element x is in a set A by $x \in A$, and that x is not in A by $x \notin A$.

For example, $A = \{1, 2, 3\}$ is a set with three elements, and by definition $1 \in A$, $2 \in A$, $3 \in A$. However, the number four is not in A , so $4 \notin A$. Now the order of the elements in a set does not matter, so $\{1, 2, 3\} = \{3, 2, 1\} = \{2, 1, 3\}$, and so on. Similarly, a set only cares about distinct members so $\{1, 2, 3, 3, 3\} = \{1, 2, 3\}$. We don't count the extra threes, as they are the same object and sets don't care for duplicates. We should also learn how to manipulate sets. For example, we can take the union of two sets A and B , denoted by $A \cup B$, which is the set of all elements that are in A or in B . In mathematics we use the word "or" in the inclusive sense, so if an element is in both A and B , it is still in $A \cup B$. This is denoted as $A \cup B = \{x : x \in A \text{ or } x \in B\}$. Similarly, we can take the intersection of two sets A and B , denoted by $A \cap B$, which is the set of all elements that are in both A and B . This is denoted as $A \cap B = \{x : x \in A \text{ and } x \in B\}$. A set can be thought of as a box which contains elements, separated by a comma which leads us to our next definition.

Definition 2 (Empty Set). The *empty set* is the unique set that contains no elements. Often denoted by $\emptyset$ or $\{\}$.

The elements in sets can be anything (ignoring ZFC for now), so we can put sets inside a set. For example $\{\emptyset\}$ is a set containing one element, the empty set. Equivalently $\{\{\}\}$ is a set with one element, the empty set (So $\{\emptyset\} = \{\{\}\}$). To give another example, the set $B = \{1, 2, 3, a, \{b, c\}\}$ is a set with 5 elements (not 6), specifically, $1 \in B$, $2 \in B$, $3 \in B$, $a \in B$, $\{b, c\} \in B$. Although B contains the element $\{b, c\}$, we should recognise that $b \notin B$ and $c \notin B$. So even if our set contains an element that is a set, and that set contains more elements inside of it, it is not considered an element of the set it is contained in. However, given a set $C = \{1, 2, 3, a, \{b, c\}, b, c\}$ of course has $b \in C$ and $c \in C$.

Now that we know what a set is, let's look at the definition of a function which will help us pair every element from a given set, to another element from a different (or the same) set.

Definition 3 (Function). Given two sets A and B , suppose that each $x \in A$ is related to one element of B , denoted by $f(x)$. Then we call f a *function* from A to B . Denoted by $f : A \rightarrow B$. Here we call A the *domain* of f , and B the *codomain* of f . Also note that the *range* is the set $\{f(x) : x \in A\}$. A function $f : A \rightarrow B$ is *injective* (or one-to-one, i.e. each input has its own unique output) if $f(a) = f(b)$ implies that $a = b$. A function $f : A \rightarrow B$ is *surjective* (or onto, i.e. every codomain (output) element is reached by at least one domain (input) element) if, for each $b \in B$ there exists some $a \in A$ such that $f(a) = b$. A function $f : A \rightarrow B$ is *bijective* if it is both injective and surjective.

Naturals, Integers, and Rationals.

Definition 4 (Natural numbers). The set of natural numbers denoted by $\mathbb{N}$ is $\{0, 1, 2, 3, \dots\}$. An element of $\mathbb{N}$ is a *natural number*.

Definition 5 (Integers). The set of integers $\mathbb{Z}$ is the set $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. An element of $\mathbb{Z}$ is an *integer*.

Definition 6 (Rationals). The set of rationals $\mathbb{Q}$ is the set

$$\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}.$$

An element of $\mathbb{Q}$ is a *rational number*.

These sets build on each other, so we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$. Ok but how do we find these relationships, and why do we care about these sets? One approach is to look at the properties of the sets and see how and why we want to fill in the "gaps". For example, $\mathbb{N}$ the natural numbers starts at zero and increments by one towards infinity. However it never actually reaches infinity. It's *closed* under the incrementation operation, but not the decrementation operation (which subtracts a given value by one unit).

Definition 7 (Closure). A set S is *closed* under an operation $*$ if for each $a, b \in S$, the result $a * b$ is also in S .

Similarly, $\mathbb{N}$ is also closed under addition and multiplication, but not subtraction or division (subtracting 3 from 2 returns -1 , which is not in $\mathbb{N}$, and dividing 2 by 3 returns $\frac{2}{3}$, which is also not in $\mathbb{N}$). So we can see that $\mathbb{N}$ is closed under some operations, but not others. If we introduce the integers $\mathbb{Z}$, we fill in the "gaps" of $\mathbb{N}$ as $\mathbb{Z}$ is closed under everything that $\mathbb{N}$ is closed under, and also closed under subtraction. However, $\mathbb{Z}$ is not closed under division (the same problem; dividing 2 by 3 returns $\frac{2}{3}$ and $\frac{2}{3} \notin \mathbb{Z}$). As you'd expect, the rationals $\mathbb{Q}$ fill in the "gaps" of $\mathbb{Z}$ as $\mathbb{Q}$ is closed under everything that $\mathbb{Z}$ is closed under, while also being closed under division (except for division by zero, which is undefined, and also why we define $q \neq 0$ in its definition). And so the set of integers contains all the natural numbers, hence $\mathbb{N} \subseteq \mathbb{Z}$, and the set of rationals contains all the integers, hence $\mathbb{Z} \subseteq \mathbb{Q}$. Leaving us with $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$.

So we have a pretty cool set to work with, the rationals $\mathbb{Q}$. Let's precisely define some of the properties of $\mathbb{Q}$. First is of course closure, under addition, subtraction, multiplication, and division. A way to write this precisely is to say if $a, b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$ (closure under addition). If $a, b \in \mathbb{Q}$, then $a - b \in \mathbb{Q}$ (closure under subtraction). If $a, b \in \mathbb{Q}$, then $ab \in \mathbb{Q}$ (closure under multiplication). If $a, b \in \mathbb{Q}$ and $b \neq 0$, then $\frac{a}{b} \in \mathbb{Q}$ (closure under division). Another interesting property is the density of the rationals.

Theorem 8 (Density). Between any two rational numbers a and b , we can always find another rational number $c := \frac{a+b}{2}$.

Proof. Let $a, b \in \mathbb{Q}$ with $a < b$. Define $c := \frac{a+b}{2}$. Since $\mathbb{Q}$ is closed under addition and division by non-zero rationals, $c \in \mathbb{Q}$.

Now,

$$c - a = \frac{a+b}{2} - a = \frac{b-a}{2} > 0$$

because $b > a$. Thus $a < c$. And,

$$b - c = b - \frac{a + b}{2} = \frac{b - a}{2} > 0,$$

Thus $c < b$. Combining both inequalities, $a < c < b$, so there exists a rational number c between a and b . □

This is cool, however it doesn't mean that $\mathbb{Q}$ is "complete". For example, $\sqrt{2} \notin \mathbb{Q}$. Why does that matter? Well there's a lot of reasons, but one of the most important is that $\mathbb{Q}$ is not closed under taking limits. For example, consider the sequence $a_n = 1, 1.4, 1.41, 1.414, 1.4142, \dots$, which is a sequence of rational numbers that converges to $\sqrt{2}$, which is not a rational number. So we have a sequence of rational numbers that converges to a non-rational number. This is a problem because we want to be able to take limits of sequences and still stay within the same set. Another problem is that $\mathbb{Q}$ is not algebraically closed. For example, the polynomial $x^2 - 2$ has no roots in $\mathbb{Q}$. So we want to "fill in the gaps" of $\mathbb{Q}$ to get a set that is closed under taking limits and algebraically closed, which leads us to the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ which we won't delve into yet. And that's it for now.